Recall: let $\sum a_n x^n$ be a power series
with radius of convergence R

Theorem (Th 23.1 in book)
The power series converges for $|x| < R$.
diverges for $|x| > R$

theorem gives us no information about resulting function
$f(x) = \sum a_n x^n \qquad |x| < R$

get better theorem
Theorem (Th. 26.4 ?) let $0 < R_1 < R$
$\Rightarrow$ The power series $\sum a_n x^n$ converges uniformly
to a continuous function $f(x)$ on $[-R_1, R_1]$

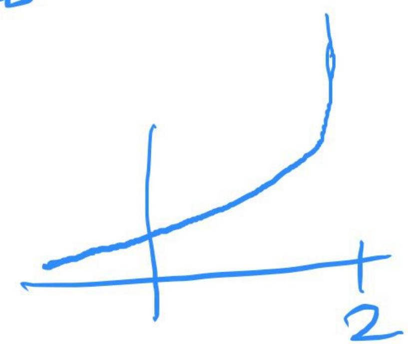
special case: Ex. $\sum_{n=0}^{\infty} 2^{-n} x^n = f(x) = \frac{2}{2-x}$. for $(-2,2)$.

Corollary: Power series $\sum a_n x^n$ converges to a cont. function on $(-R, R)$

Remark: convergence in corollary may or may not be uniform.

check: convergence of $\sum_{n=0}^{\infty} 2^{-n} x^n$ to $\frac{2}{2-x}$ NOT uniform

on $(2, -2)$



$\leftarrow y = \frac{2}{2-x}$

next. study properties of differentiability and integrability of function $\boxed{f(x) = \sum_{n=0}^{\infty} a_n x^n}$

We will assume properties of integrals and differentiability before studying them rigorously need following lemma:

Lemma If $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R

$\Rightarrow$ $\sum_{n=1}^{\infty} n a_n x^{n-1}$ and $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$

also have radius of convergence R

Proof Observe:

$$\underbrace{x \cdot \sum_{n=1}^{\infty} n a_n x^{n-1}}_{\text{series converges}} = \underbrace{\sum_{n=1}^{\infty} n a_n x^n}_{\text{series converges}}$$

series converges $\Leftrightarrow$ series converges

$\Rightarrow$ enough to show that R_1 = radius of convergence of $\sum n a_n x^n$

$= R =$ " " " " $\sum a_n x^n$

Recall: $\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}$

$\frac{1}{R_1} = \limsup_{n\to\infty} |n a_n|^{1/n} = \underbrace{\lim_{n\to\infty} n^{1/n}}_{=1} \cdot \underbrace{\limsup |a_n|^{1/n}}_{=1/R}$

$= 1/R$

to show: $\frac{1}{R} = \frac{1}{R_1}$!

Claim: $\boxed{\lim_{n\to\infty} n^{1/n} = 1}$

(was proved in 142A

quick proof using calculus:

observe $x^{1/x} = e^{(\ln x)/x}$

Apply l'Hospital to exponent:

$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0 \Rightarrow \lim_{n\to\infty} n^{1/n} = 1$

One similarly shows that

$$\limsup \left(\frac{|a_n|}{n+1}\right)^{1/n} = \limsup |a_n|^{1/n}$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \quad \text{has radius of convergence } R$$

We are going to use the following properties of the integral (to be proved later):

- $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
- $\displaystyle\int_a^b \sum a_n g_n(x)\,dx = \sum a_n \int_a^b g_n(x)\,dx$

Theorem Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R

$\Rightarrow$ its antiderivative $F(x) = \int_0^x f(t)\,dt$, $|x| < R$

is given by the power series

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \quad \text{for } |x| < R$$

i.e. we can integrate the power series term by term.

Proof. do case $x < 0$.

$$\int_x^0 f(t)\,dt = \int_x^0 \lim_{n\to\infty} \left(\sum_{k=0}^{n} a_k t^k\right) dt$$

$$= \lim_{n\to\infty} \int_x^0 \sum_{k=0}^{n} a_k t^k \,dt \quad \text{(by uniform conv.}$$

property of integrals

$$= \lim_{n\to\infty} \sum_{k=0}^{n} a_k \int_x^0 t^k \, dt$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} a_k \left[\cancel{\frac{0^{k+1}}{k+1}} - \frac{x^{k+1}}{k+1} \right]$$

$$= - \sum a_k \frac{x^{k+1}}{k+1}$$

$$\Rightarrow \int_0^x f(t)\, dt = - \int_x^0 f(t)\, dt = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$$

Remark: We will show that
$$f'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}$$
i.e. we can differentiate term by term.

But the proof will be more complicate because of

Observation $f_n \to f$ uniformly
does NOT imply $f_n' \to f'$

Example: Let $f_n(x) = \frac{1}{n} \sin nx$
have seen $f_n \to 0$ (zero function) uniformly

BUT: $f_n'(x) = \frac{1}{n} \cos nx \cdot n = \cos nx$

$f_n'(x)$ usually does not converge, not even pointwise!

e.g. $x = \pi$: $\cos n\pi = (-1)^n \Rightarrow (\cos n\pi)_n = ((-1)^n)_n$ does not converge!